Quantum States in the Multipole Solution of the Maxwell's Equations for the Free Radiation Field

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Abstract

Classical electrodynamics built on the multipole approach to the Maxwell's theory of light for the empty space has the potential for reproducing fundamental aspects of quantum optics. Field quantization and discrete energy levels are found without the use of the correspondence principle that is fundamental to the conventional connection between the classical field modes and quantum harmonic oscillators. After a brief summary of the conventional quantum approach based on plane waves, results of the classical multipole approach are given for Fock states, zero-point energy and photon statistics of chaotic light.

Common starting point: Maxwell's equations for the free radiation field (Classical theory of light)

$$\nabla\cdot\mathbf{E}(\mathbf{r},t)=0$$
$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0$$
$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}$$
$$\nabla\times\mathbf{B}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}$$

Wave equation

$$\nabla^2\mathbf{E}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2}\qquad \nabla^2\mathbf{B}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial^2\mathbf{B}(\mathbf{r},t)}{\partial t^2}$$

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)\qquad \mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

Wave equation

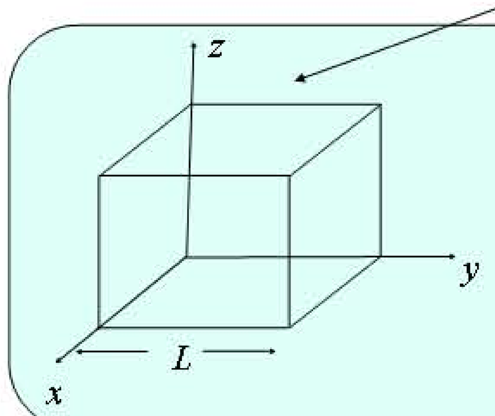
$$\nabla^2\mathbf{A}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2}$$

Classical energy $\mathscr{E}=\frac{1}{2}\varepsilon_0\int d\mathbf{r}\left|\mathbf{E}(\mathbf{r},t)\right|^2+\frac{1}{2\mu_0}\int d\mathbf{r}\left|\mathbf{B}(\mathbf{r},t)\right|^2$

It is common knowledge that the quantization is an impossible task within the classical theory of light and it comes as no surprise to read that "the energy eigenvalues are discrete, in contrast to classical electromagnetic theory" [pag. 10 in M. O. Scully, M. S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge, 1999], or "zero-point energy or vacuum energy… has no analogue in the classical theory" [pag. 143 of R. Loudon, The Quantum Theory of Light, Oxford University Press, Oxford, 2000]. Whether this is the ultimate truth is open to debate according to the results of the current work inspired by the classical multipole expansion of the electromagnetic field [J. D. Jackson, Classical Electrodynamics, Wiley, New York, 1999].

Canonical quantization: plane-wave approach + Bohr's correspondence principle (Quantum theory of light)

"There is no way to make up your mind whether the electromagnetic field is really to be described as a quantized harmonic oscillator or by giving how many photons there are in each condition. The two views turn out to be mathematically identical. So in the future we can speak either about the number of photons in a particular state in a box or the number of energy level associated with a particular mode of oscillation of the electromagnetic field. They are two ways of saying the same thing. The same is true of photons in free space. They are equivalent to oscillations of a cavity whose walls have receded to infinity." (The Feynman Lectures on Physics, Vol. 3, page 4-9, section 4-5)



$$\frac{L}{\lambda}>>1\longleftrightarrow\frac{V}{\lambda^3}>>1\qquad \sum_{\mathbf{k},s}=2\sum_{\mathbf{k}}\rightarrow\frac{V}{\pi^2}\int k^2dk$$

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k},s}A_{\mathbf{k},s}(\mathbf{r},t)\,\mathbf{e}_{\mathbf{k},s}\qquad A_{\mathbf{k},s}(\mathbf{r},t)=A^0_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}+c.c.$$

Classical energy according to the plane-wave approach

$$\mathscr{E}=2\varepsilon_0 V\sum_{\mathbf{k},s}\omega_k^2\left|A^0_{\mathbf{k},s}\right|^2$$

Conversion established by the correspondence principle

$$A^0_{\mathbf{k},s}\rightarrow\left(\frac{\hbar}{2\varepsilon_0 V\omega_k}\right)^{1/2}\hat{a}_{\mathbf{k},s}\qquad A^{0*}_{\mathbf{k},s}\rightarrow\left(\frac{\hbar}{2\varepsilon_0 V\omega_k}\right)^{1/2}\hat{a}^+_{\mathbf{k},s}$$

Field operators acting on Fock (photon-number) states

$$\hat{\mathbf{A}}(\mathbf{r},t)=\sum_{\mathbf{k},s}\hat{A}_{\mathbf{k},s}(\mathbf{r},t)\,\mathbf{e}_{\mathbf{k},s}\qquad \hat{A}_{\mathbf{k},s}(\mathbf{r},t)=\left(\frac{\hbar}{2\varepsilon_0 V\omega_k}\right)^{1/2}\left\{\hat{a}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}+\hat{a}^+_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\right\}$$

$$\hat{\mathbf{E}}(\mathbf{r},t)=\sum_{\mathbf{k},s}\hat{E}_{\mathbf{k},s}(\mathbf{r},t)\,\mathbf{e}_{\mathbf{k},s}\qquad \hat{E}_{\mathbf{k},s}(\mathbf{r},t)=\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\left\{\hat{a}_{\mathbf{k},s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t+\pi/2)}+h.c.\right\}$$

$$\hat{\mathbf{B}}(\mathbf{r},t)=\sum_{\mathbf{k},s}\hat{B}_{\mathbf{k},s}(\mathbf{r},t)\,\mathbf{k}\times\mathbf{e}_{\mathbf{k},s}\qquad \hat{B}_{\mathbf{k},s}(\mathbf{r},t)=\left(\frac{\hbar}{2\varepsilon_0\omega_k V}\right)^{1/2}\left\{\hat{a}_{\mathbf{k},s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t+\pi/2)}+h.c.\right\}$$

Quantum Hamiltonian (series of harmonic oscillators) $\hat{H}=\sum_{\mathbf{k},s}\hbar\omega_k\left(\hat{a}^+_{\mathbf{k},s}\hat{a}_{\mathbf{k},s}+\frac{1}{2}\right)$

Quantum energy (single mode) $\varepsilon_{\mathbf{k},s}=\hbar\omega_k\left(n_{\mathbf{k},s}+\frac{1}{2}\right)$

Quantum states in classical electrodynamics based on multipole expansion (spherical functions)

Step 1
the energy

Definition: $<>=$ cycle average

Mode decomposition $\mathbf{E}(\mathbf{r},t)=\sum_{\mathbf{k},s}E_{\mathbf{k},s}(\mathbf{r},t)\,\mathbf{e}_{\mathbf{k},s}$

$$\mathscr{E}=\frac{1}{2}\varepsilon_0\int d\mathbf{r}<\left|\mathbf{E}(\mathbf{r},t)\right|^2>+\frac{1}{2\mu_0}\int d\mathbf{r}<\left|\mathbf{B}(\mathbf{r},t)\right|^2>=\varepsilon_0\int d\mathbf{r}<\left|\mathbf{E}(\mathbf{r},t)\right|^2>=\varepsilon_0\sum_{\mathbf{k},s}\int d\mathbf{r}<\left|E_{\mathbf{k},s}(\mathbf{r},t)\right|^2>$$

Step 2
Scalar wave equation and Helmholtz equation for the mode field

$$\nabla^2E_{\mathbf{k},s}(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial^2E_{\mathbf{k},s}(\mathbf{r},t)}{\partial t^2}\xrightarrow{\text{Harmonic hypothesis}}$$

Helmholtz equation:
$$\nabla^2E_{\mathbf{k},s}(\mathbf{r},t)+k^2E_{\mathbf{k},s}(\mathbf{r},t)=0$$

Spherical polar coordinates; Separation of variables

$$E_{\mathbf{k},s}(\mathbf{r},t)=E_{0,s}(t)\psi_{\mathbf{k},s}(\mathbf{r})\qquad \psi_{\mathbf{k},s}(\mathbf{r})=R_{\mathbf{k},s}(r)\Omega_{\mathbf{k},s}(\vartheta,\varphi)$$

$$\frac{1}{R_{\mathbf{k},s}}\frac{d}{dr}\left(r^2\frac{dR_{\mathbf{k},s}}{dr}\right)+k^2r^2=-\frac{1}{sin\vartheta\,\Omega_{\mathbf{k},s}}\frac{\partial}{\partial\vartheta}\left(sin\vartheta\frac{\partial\Omega_{\mathbf{k},s}}{\partial\vartheta}\right)-\frac{1}{(sin\vartheta)^2\Omega_{\mathbf{k},s}}\frac{\partial^2\Omega_{\mathbf{k},s}}{\partial\varphi^2}$$

Step 3
Solution of the radial and angular components of the Helmholtz equation

Radial equation:
$$\frac{d}{dr}\left(r^2\frac{dR_{\mathbf{k},s}}{dr}\right)+\left[k^2r^2-n(n+1)\right]R_{\mathbf{k},s}=0\longrightarrow R_{\mathbf{k},s}(r)=j_n(kr)$$

Spherical Bessel function

$$\iint d\vartheta d\varphi\, sin\vartheta\, Y^{m'*}_{n'}(\vartheta,\varphi)Y^m_n(\vartheta,\varphi)=\delta_{n,n'}\delta_{m,m'}$$

Orthonormality

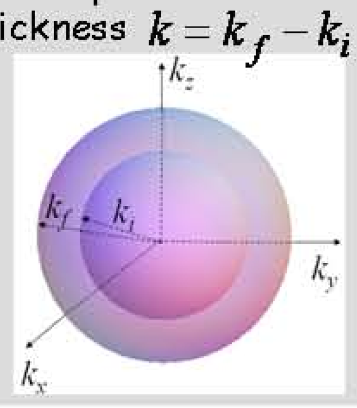
Angular equation:
$$-\frac{1}{sin\vartheta\,\Omega_{\mathbf{k},s}}\frac{\partial}{\partial\vartheta}\left(sin\vartheta\frac{\partial\Omega_{\mathbf{k},s}}{\partial\vartheta}\right)-\frac{1}{(sin\vartheta)^2\Omega_{\mathbf{k},s}}\frac{\partial^2\Omega_{\mathbf{k},s}}{\partial\varphi^2}=n(n+1)\longrightarrow \Omega_{\mathbf{k},s}(\vartheta,\varphi)=\sum_{m=-n}^{n}e^{i\alpha_{n,m}}Y^m_n(\vartheta,\varphi)$$

Spherical harmonics

Step 4
Quantized energy and quantized field components

Solution of order n: $E^{(n)}_{\mathbf{k},s}(\mathbf{r},t)=E_{0,s}(t)\,j_n(kr)\sum_{m=-n}^{n}e^{i\alpha_{n,m}}Y^m_n(\vartheta,\varphi)$

Definition: Energy associated with the solution of order n for a spherical shell of thickness $k=k_f-k_i$



$$\mathscr{E}^{(n)}_{k=k_f-k_i}=\varepsilon_0\sum_{\mathbf{k}'}\int d\mathbf{r}<\left|E^{(n)}_{\mathbf{k}',s}(\mathbf{r},t)\right|^2>=\varepsilon_0\frac{|E_0|^2}{2}\frac{V}{2\pi^2}\int_{k_i}^{k_f}dk'\,k'^2(2n+1)R_n(k')=\eta\omega_k\left(n+\frac{1}{2}\right)$$

Quantized classical energy with vacuum contribution !!!

$$\eta=\frac{\varepsilon_0}{4\pi^2c}RV|E_0|^2$$

$$<\left|E_{0,s}(t)\right|^2>=\frac{|E_0|^2}{2}\qquad R_n(k')=\int_0^R dr\,r^2\,j_n^2(k'r)\xrightarrow{k'R>>1}\frac{R}{2k'^2}$$

$$\sum_{\mathbf{k}'}=\sum_{\mathbf{k}'}\rightarrow\frac{V}{2\pi^2}\int_{k_i}^{k_f}dk'\,k'^2\qquad \sum_{m,m'=-n}^{n}e^{i(\alpha_{n,m}-\alpha_{n,m'})}\iint d\vartheta d\varphi\,sin\vartheta\,Y^{m'*}_n(\vartheta,\varphi)Y^m_n(\vartheta,\varphi)=2n+1$$

Step 5
Fock states and chaotic light

Definition of probability distribution P_n associated with the energy level n implies that $\mathscr{E}_{aver}=\sum_nP_n\mathscr{E}^{(n)}_k$

Fock state of the single mode $(\mathbf{k},s)$
$$\begin{cases}P_{n_{\mathbf{k},s}}=1 & \text{if } n_{\mathbf{k},s}=n\\ P_{n_{\mathbf{k},s}}=0 & \text{if } n_{\mathbf{k},s}\neq n\end{cases}\Rightarrow\mathscr{E}_{aver}=\mathscr{E}^{(n)}_k=\eta\omega_k\left(n+\frac{1}{2}\right)$$

Boltzmann probability distribution:
$$P_n=e^{-\frac{\mathscr{E}^{(n)}_k}{k_BT}}\Big/\sum_ne^{-\frac{\mathscr{E}^{(n)}_k}{k_BT}}\Rightarrow\begin{cases}\mathscr{E}_{aver}=\eta\omega_k\left(<<n>>+\frac{1}{2}\right)\\ <<n>>=\sum_nP_n\,n=1/\left(e^{\eta\omega_k/k_BT}-1\right)\end{cases}$$

Chaotic or thermal light