Simple derivation of Schrödinger equation from Newtonian dynamics

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Abstract

If Schrödinger representation of quantum mechanics reproduces Newton's laws of motion in terms of expectation values (Ehrenfest theorem), the contrary is considered elusive. Against this opinion, we present here a simple method to make Newtonian dynamics develop into Schrödinger representation. The proof is laid out in two steps. First, Newton's laws of motion are used to determine a classical wave equation whose similarity with the Schrödinger equation is mediated by a parameter that plays the identical role of the constant K introduced by Schrödinger in the original formulation of his theory. In the second step, the classical wave equation becomes exactly the Schrödinger equation thanks to the numerical value of the parameter obtained from the identification of the classical momentum with de Broglie momentum of matter waves.

I.   INTRODUCTION

One of the main routes to quantum mechanics runs through the Schrödinger equation and the concept of wave function.[1] Despite the undoubted importance of this cornerstone of modern physics, the subject is ordinarily introduced in courses on quantum mechanics without much detail about its conceptual foundations, with the result that, according to prominent physicists,[2] the derivation of the Schrödinger equation conveys a sense of dissatisfaction. In reality, the disappointment has something to do with the heuristic explanation reported by Schrödinger in his original conjecture.[3] Its weakness is, for instance, underlined by Feynman in the third volume of the Lectures where one can read the following comment about the founder of wave mechanics: "some of the arguments he used were even false, but that does not matter; the only important thing is that the ultimate equation gives a correct description of nature".[4] Against this compliant attitude, Feynman himself provided us with a derivation of the Schrödinger equation that led him to the path integral formulation of quantum field theory.[5, 6] Responding to the same stimulus and in the hope of a sound explanation of the equation, many other physicists have come up with several proposals that span the last 50 years.[2, 7-23] Along this line of research, the current attempt aims at the definition of a simple and, at the same time, rigorous approach to the task. But, before going into the details and to better contextualize our purpose, let us make some preliminary remarks concerning ordinary approaches to the Schrödinger picture of quantum mechanics.

In general, the introduction to this fundamental subject of modern physics is made on the basis of three mutually exclusive didactic criteria. The simplest responds to the legitimate claim that the Schrödinger equation is so well known that its detailed derivation can be neglected in favor of its solutions to fundamental problems.[24, 25] Under this orientation, the equation is simply assumed as a fact. On the other hand, the undisputable demand for a deeper physical insight urges us to provide a theoretical minimum at least.[26] To this end, the most

common approach draws its inspiration from plausibility arguments[27-29] that are simpler analogs of the original idea developed by Schrödinger. Beyond their undeniable strength of pedagogical interest, such proposals are not without shortcomings (e.g., restriction to the free particle, assumption of a constant potential) and, for this reason, they are not good enough for whoever is in pursuit of a flawless method to secure the Schrödinger equation against any doubt about its derivation. From this perspective, a considerable number of authors has met the challenge of an exact formulation from first principles.[2, 7-23] Among these attempts, the Hamilton-Jacobi theory of classical mechanics has attracted a lot of attention because of the guidance it gave to Schrödinger in the discovery of his equation. Unlike this acknowledged theoretical standpoint, which presents itself with nonlinearities that must be suppressed in some manner (see, for instance, the discussion in Ref. 2), here we take the alternative view that the Schrödinger equation is directly achievable through Newton's laws of dynamics.[7, 13]

The connection was clear soon after the birth of modern quantum mechanics (Ehrenfest theorem)[30] and, nowadays, it is common knowledge that the Schrödinger equation "plays a role logically analogous to Newton's second law".[25] Thus, the objective of this work is to prove that the correspondence is stronger than an analogy. However, it must be remarked that the current proof has the fundamental premise in the de Broglie postulate of matter waves. In this regard, our starting point coincides with the assumption from which Schrödinger moved in search of the equation that describes waves peculiar to massive particles.[3] In addition, the postulate has the further benefit of avoiding the stochastic forces that characterize former proposals made to show that the radical departure from classical physics is unnecessary.[7, 13]

The work is organized as follows. First of all, in Section II, we describe how the second law of Newtonian dynamics can generate a classical wave equation to be solved for a function that takes into account the loss of information about the dynamical state of the particle. In Section III, considering the reference to the Ehrenfest theorem, the classical wave equation is

applied to the case of conservative forces and, thanks to the divergence theorem of three-dimensional calculus, the Schrödinger equation is recovered. The final Section IV is relative to the conclusions.

II. DYNAMICAL LAW OF A NEWTONIAN PARTICLE WITHOUT ACCURATE KNOWLEDGE OF ITS MOMENTUM

The approach considered here begins with the introduction of a three-dimensional Cartesian reference frame within which a classical particle of mass m follows a trajectory given by the vector $\mathbf{r}(t)$ depending on the time t. If we were to stick to the program of classical mechanics, the trajectory should be complemented by the speed $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ or the associated momentum $\mathbf{p}(t) = m\mathbf{v}(t)$ subject to the dynamical law $d\mathbf{p}/dt = m\ddot{\mathbf{r}} = \mathbf{F}$ with $\mathbf{F}$ the net force acting on the particle. But, let us suppose that, for some reason, it is impossible to get accurate information about $\mathbf{v}(t)$ or $\mathbf{p}(t)$ while measuring $\mathbf{r}(t)$ at certain specific times or, more generally, we disregard this sort of information. The hypothesis bears a manifest resemblance to the quantum-mechanical uncertainty principle established by Heisenberg,[24-29] but we ignore it in agreement with our classical framework. We also avoid the conceptual scheme behind other classical attempts that, equally based on Newtonian dynamics, give reason for the uncertainty to exist on the account of an universal Brownian motion resulting from diffusion processes with coefficient $\hbar/(2m)$ and no friction.[7, 13] Nonetheless, skipping any investigation on possible causes for such a failure of traditional determinism on which classical mechanics is grounded, we will take the uncertainty for granted and treat it as described below. Given this scenario, it is then legitimate to ask whether we can still collect dynamical information about the particle.

To answer the question, we introduce an arbitrary function $f(\xi)$ whose argument ξ can be related to any specific value $\overline{\mathbf{p}}$ of the momentum. It is impossible to tell when the particle momentum $\mathbf{p}(t)$ takes the chosen value of $\overline{\mathbf{p}}$, but we need the function $f(\xi)$ that could store the information about the particle traveling with momentum $\overline{\mathbf{p}}$ during its trajectory $\mathbf{r}(t)$. We do not know yet how this can be done, but we set off on a hunt for a solution. A promising starting point is to assume that the argument ξ is dimensionless and written as the scalar product between $\overline{\mathbf{p}}$ and $\mathbf{r}(t)$, that is $\xi = \alpha \, \overline{\mathbf{p}} \cdot \mathbf{r}$ with α a proportionality constant that takes into account the physical units of the scalar product. This particular choice of the variable ξ is convenient because, as soon as we consider the evolution of $f(\xi)$, its time derivative depends on the time derivative of ξ which, in turn, is connected to the kinetic energy $|\overline{\mathbf{p}}|^2 / (2m)$ when $\dot{\mathbf{r}} = \overline{\mathbf{v}} = \overline{\mathbf{p}} / m$. Shortly, we are going to see how this information can be turned to our advantage. In addition, the role played here by the constant α is identical to the role given by Schrödinger to the constant K appearing in the Hamilton principal function $S = K \log \psi$ with ψ the later known Schrödinger wave function [see Eq. (2) in the first paper of Ref. 3] and, in complete analogy with the use of K made by the founder of wave mechanics, we will treat our constant α as a free parameter whose value will be determined by the evidence of matter waves with momenta given by the de Broglie equation.

Although the function $f(\xi)$ has been introduced without any specific definition of its properties (except for the above-mentioned definition of ξ), some conditions restrict the choice. First of all, its evolution depends on the trajectory $\mathbf{r}(t)$ extracted from the second Newton's law $m\ddot{\mathbf{r}} = \mathbf{F}$. To determine this evolution, we can calculate the first two time derivatives of $f(\xi)$

$$\frac{df(\xi)}{dt} = \alpha \frac{df(\xi)}{d\xi} \frac{d}{dt}(\bar{\mathbf{p}} \cdot \mathbf{r}) = \alpha \bar{\mathbf{p}} \cdot \dot{\mathbf{r}} \frac{df(\xi)}{d\xi} \tag{1}$$

$$\frac{d^2 f(\xi)}{dt^2} = \alpha^2 (\bar{\mathbf{p}} \cdot \dot{\mathbf{r}})^2 \frac{d^2 f(\xi)}{d\xi^2} + \alpha \bar{\mathbf{p}} \cdot \ddot{\mathbf{r}} \frac{df(\xi)}{d\xi} \tag{2}$$

and considering that

$$\nabla f(\xi) = \alpha \, \bar{\mathbf{p}} \frac{df(\xi)}{d\xi} \tag{3}$$

$$\nabla^2 f(\xi) = \alpha^2 \, \bar{p}^2 \frac{d^2 f(\xi)}{d\xi^2} \tag{4}$$

with $\bar{p}^2 = \sum_i \bar{p_i}^2 = |\bar{\mathbf{p}}|^2$, we find

$$-\frac{(\bar{\mathbf{p}} \cdot \dot{\mathbf{r}})^2}{\bar{p}^2} \nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2} = \alpha \bar{\mathbf{p}} \cdot \ddot{\mathbf{r}} \frac{df(\xi)}{d\xi} \tag{5}$$

which establishes the dynamical law followed by the function $f(\xi)$ in order to carry the information about the state of the particle. The significance of Eq. (5) stems from the left-hand side where we have the description of a wave with an associated propagation speed of $(\bar{\mathbf{p}} \cdot \dot{\mathbf{r}})^2 / \bar{p}^2$. The right-hand side can be regarded as the source term of the wave and contains, instead, the particle-like feature arising from the scalar product $\bar{\mathbf{p}} \cdot \ddot{\mathbf{r}}$. Remarkably, Eq. (5) does not show an explicit dependence on the mass. This one is incorporated in the chosen momentum $\bar{\mathbf{p}}$ and, in principle, Eq. (5) works equally well for massless particles for which it is possible to define a momentum. Regardless of these details, we recover the known

homogenous wave equation for free particles being $\ddot{\mathbf{r}} = 0$ and hence the right-hand side of Eq. (5) vanishing, that is

$$-\frac{\left(\overline{\mathbf{p}} \cdot \dot{\mathbf{r}}\right)^2}{\overline{p}^2} \nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2} = 0 . \tag{6}$$

In view of its importance, this case is treated in the remaining part of this Section and Eq. (6) reduces to

$$\nabla^2 f(\xi) - \frac{1}{|\overline{\mathbf{v}}|^2} \frac{d^2 f(\xi)}{dt^2} = 0 \tag{7}$$

where we have used the fact that the momentum the free particle had at the beginning is conserved and $\dot{\mathbf{r}} = \overline{\mathbf{p}} / m = \overline{\mathbf{v}}$ at any time. Actually, Eq. (7) differs from the familiar wave equation by the total time derivative that replaces the more usual partial derivative. Although the total derivatives appears in classical mechanics of waves in continuous media,[31] this little mismatch is only apparent. Indeed, considering that $\xi = \alpha \overline{\mathbf{p}} \cdot \mathbf{r} = \alpha \overline{\mathbf{p}} \cdot \dot{\mathbf{r}} t = \alpha \overline{\mathbf{p}} \cdot \overline{\mathbf{v}} t = \Omega t$ with Ω a constant angular frequency, the identity

$$\frac{df(\xi)}{dt} = \frac{\partial f(\xi)}{\partial t} \tag{8}$$

is trivially satisfied for the free particle. Furthermore, the time derivative of ξ results in

$$\dot{\xi} = \alpha \overline{\mathbf{p}} \cdot \dot{\mathbf{r}} = \alpha \overline{p} \overline{v} = k \overline{v} = \Omega \tag{9}$$

where we have made use of the definition $k = \alpha \bar{p}$. As expected, this parameter has the physical dimension of a wave vector and, if we introduce a wavelength λ such that $\Omega = 2\pi \bar{v}/\lambda$, then we find $k = 2\pi/\lambda$ and

$$\bar{p} = \frac{2\pi}{\alpha \lambda}. \tag{10}$$

The last equation expresses the result that the momentum of a free particle is inversely proportional to the wavelength characterizing the wave function solution to Eq. (7). The result of Eq. (10) was expected and, although echoes de Broglie relationship between momentum and wavelength of a matter wave, is entirely classical. The quantum nature of Eq. (10) is instead manifest as soon as the exact correspondence to the de Broglie relationship is imposed and, at this point, the parameter α equals the inverse of the reduced Planck constant $\hbar = h/2\pi$. This identification will be used at the end of the next Section.

III. CONSERVATIVE FORCES: FROM NEWTON TO SCHRÖDINGER

The idea of an intimate relationship between Newton's laws of dynamics and Schrödinger equation dates back to 1927 when Ehrenfest came out with the well-known theorem that bears his name.[30] The theorem states that, given a quantum state represented by the Schrödinger wave function, then the expectation value of the time derivative of the momentum operator is equal to the expectation value of the negative gradient of the potential energy function. In other words, the quantum-mechanical expectation values reproduce the structure of the second Newton's law for conservative forces. The current attempt aims, instead, at a complete role reversal between hypothesis and thesis of the Ehrenfest theorem.

Indeed, given Newton's laws, we try to derive the Schrödinger equation. The underlying assumption is the de Broglie momentum that establishes the necessary premise for the Schrödinger theory as well as the current work.

To fulfill the plan, we need to prove that Eq. (5) suits our purposes when the second Newton's law reads

$$\ddot{\mathbf{r}} = -\frac{\nabla U}{m} \tag{11}$$

where the potential energy $U = U(\mathbf{r})$ is a function of the spatial coordinates only. The combination of Eqs. (5) and (11) is open to further treatment after the multiplication by $f(\xi)$ and three-dimensional spatial integration

$$\int f(\xi)\left[-\frac{(\overline{\mathbf{p}} \cdot \dot{\mathbf{r}})^2}{\overline{p}^2}\nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2}\right]d\mathbf{r} = -\frac{\alpha}{m}\int[\overline{\mathbf{p}} \cdot \nabla U(\mathbf{r})]f(\xi)\frac{df(\xi)}{d\xi}d\mathbf{r}. \tag{12}$$

Next, we apply the divergence theorem (or Gauss theorem)[32] useful for the integration by parts. This version is obtained when the theorem is applied to the product between a scalar function u and a vector field $\mathbf{w}$, then

$$\int_V (\mathbf{w} \cdot \nabla u)d\mathbf{r} + \int_V u(\nabla \cdot \mathbf{w})d\mathbf{r} = \int_\Gamma u(\mathbf{w} \cdot \mathbf{n})d\Gamma \tag{13}$$

where $\mathbf{n}$ is a unit vector orthogonal to the surface Γ that contains the volume V. When extended to our case, Eq. (13) allows us to rewrite the integral appearing on the right-hand side of Eq. (12) according to

$$\int_V (\overline{\mathbf{p}} \cdot \nabla U) f(\xi) \frac{df(\xi)}{d\xi} d\mathbf{r} = \int_\Gamma U(\mathbf{r}) f(\xi) \frac{df(\xi)}{d\xi} (\overline{\mathbf{p}} \cdot \mathbf{n}) d\Gamma - \int_V U(\mathbf{r}) \left\{ \nabla \cdot \left[\overline{\mathbf{p}} \, f(\xi) \frac{df(\xi)}{d\xi} \right] \right\} d\mathbf{r} \quad (14)$$

and, on condition that the surface integral is vanishing when we let the surface Γ go to infinite,

$$\int_\Gamma U(\mathbf{r}) f(\xi) \frac{df(\xi)}{d\xi} (\overline{\mathbf{p}} \cdot \mathbf{n}) d\Gamma \to 0 \quad (15)$$

then Eq. (12) can be recast as follows

$$\int f(\xi) \left[-\frac{(\overline{\mathbf{p}} \cdot \dot{\mathbf{r}})^2}{\overline{p}^2} \nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2} \right] d\mathbf{r} = \frac{\alpha^2}{m} \int U(\mathbf{r}) \overline{p}^2 \frac{d}{d\xi} \left[f(\xi) \frac{df(\xi)}{d\xi} \right] d\mathbf{r} \quad (16)$$

where we have used the identity

$$\nabla \cdot \left[\overline{\mathbf{p}} \, f(\xi) \frac{df(\xi)}{d\xi} \right] = \alpha \overline{p}^2 \frac{d}{d\xi} \left[f(\xi) \frac{df(\xi)}{d\xi} \right]. \quad (17)$$

The result of Eq. (16) is obvious if we let the two kernels be the same or

$$f(\xi) \left[-\frac{(\overline{\mathbf{p}} \cdot \dot{\mathbf{r}})^2}{\overline{p}^2} \nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2} \right] = \frac{\alpha^2 \overline{p}^2}{m} U(\mathbf{r}) \frac{d}{d\xi} \left[f(\xi) \frac{df(\xi)}{d\xi} \right] \quad (18)$$

and, now, further elaboration of Eq. (18) deserves more explaining. Indeed, additional handling of Eq. (18) has to be made in view of some conditions that could suggest the right dependences expected for the wave function $f(\xi)$. For instance, we can play with the time t that is an independent variable of our problem and, in particular, we can constrain Eq. (18) to hold for those times t_j (with $j = 0, 1, 2,...$) such that the condition $\dot{\mathbf{r}} = \bar{\mathbf{p}}/m = \bar{\mathbf{v}}$ occurs. Then, Eq. (18) reduces to

$$\left\{ f(\xi) \left[-\frac{(\bar{\mathbf{p}} \cdot \dot{\mathbf{r}})^2}{\bar{p}^2} \nabla^2 f(\xi) + \frac{d^2 f(\xi)}{dt^2} \right] \right\}\Bigg|_{t=t_j} = \frac{\alpha^2 \bar{p}^2}{m} \left\{ U(\mathbf{r}) \frac{d}{d\xi} \left[f(\xi) \frac{df(\xi)}{d\xi} \right] \right\}\Bigg|_{t=t_j} \qquad (19)$$

where the reference to the chosen times t_j has been made explicit. The constraint has important consequences. First, the derivatives appearing in the right-hand side of Eq. (19) can be rearranged if we write $d\xi = \alpha\bar{\mathbf{p}} \cdot d\mathbf{r} = \alpha\bar{\mathbf{p}} \cdot \bar{\mathbf{v}}\, dt$ that results easily from the condition $\dot{\mathbf{r}} = \bar{\mathbf{p}}/m = \bar{\mathbf{v}}$. In this way, setting $\Omega = \alpha\bar{p}^2/m$, we find

$$\left\{ \frac{d}{d\xi} \left[f(\xi) \frac{df(\xi)}{d\xi} \right] \right\}\Bigg|_{t=t_j} = \frac{1}{\Omega^2} \left\{ \left(\frac{df(\xi)}{dt}\Bigg|_{t=t_j} \right)^2 + \left[f(\xi) \frac{d^2 f(\xi)}{dt^2} \right]\Bigg|_{t=t_j} \right\} \qquad (20)$$

and Eq. (19) becomes

$$f(\xi)\big|_{t=t_j}\left\{-\frac{(\bar{\mathbf{p}}\cdot\bar{\mathbf{v}})^2}{\bar{p}^2}\left[\nabla^2 f(\xi)\right]_{t=t_j}+\frac{d^2 f(\xi)}{dt^2}\bigg|_{t=t_j}\right\}=$$

$$\frac{\alpha^2\bar{p}^2}{m\Omega^2}U(\mathbf{r})\big|_{t=t_j}\left\{\left(\frac{df(\xi)}{dt}\bigg|_{t=t_j}\right)^2+\left[f(\xi)\frac{d^2 f(\xi)}{dt^2}\right]_{t=t_j}\right\}$$

(21)

The second effect of the constraint $t=t_j$ is on the time derivatives. These can be calculated if we point out that the momentum appears to obey the conservation law typical of the free particle mentioned at the end of the previous Section. In other terms, the function $f(\xi)$ has to guarantee that the momentum returns to the chosen $\bar{\mathbf{p}}$ value at any t_j. The condition implies a restriction on the time dependence of the wave function $f(\xi)$ that should evolve according to a harmonically varying function between one t_j and the other. This means that the time derivatives can now be calculated

$$\frac{df(\xi)}{dt}\bigg|_{t=t_j}=-i\Omega f(\xi)\big|_{t=t_j}$$

(22)

$$\frac{d^2 f(\xi)}{dt^2}\bigg|_{t=t_j}=-\Omega^2 f(\xi)\big|_{t=t_j}$$

(23)

where the angular frequency Ω appears because of the time derivative of the variable ξ at $t=t_j$ as reported in Eq. (9). In the end, Eq. (21) can be transformed into

$$\frac{\bar{p}^2}{m^2}\left[\nabla^2 f(\xi)\right]_{t=t_j}+i\Omega\frac{df(\xi)}{dt}\bigg|_{t=t_j}=2\frac{\alpha^2\bar{p}^2}{m}\left[U(\mathbf{r})f(\xi)\right]_{t=t_j}$$

(24)

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and recalling that $\Omega = \alpha \bar{p}^2 / m$, we find

$$-\frac{1}{2\alpha^2 m}\left[\nabla^2 f(\xi)\right]_{t=t_j} + \left[U(\mathbf{r})f(\xi)\right]_{t=t_j} = i\frac{1}{2\alpha}\frac{df(\xi)}{dt}\bigg|_{t=t_j} \tag{25}$$

This equation is correct if we look for solutions that are independent from the choice of the reference time t_j and, by relaxing the constraint, we end up with the following result

$$-\frac{1}{2\alpha^2 m}\nabla^2 f(\xi) + U(\mathbf{r})f(\xi) = i\frac{1}{2\alpha}\frac{df(\xi)}{dt} \tag{26}$$

which is in close resemblance to the Schrödinger equation. The similarity is remarkable because Eq. (26) has been derived on the basis of classical arguments and shows many of the features that characterize the Schrödinger equation. What is more, the similarity becomes much clearer if we consider the de Broglie relationship that was used at the end of Section II to reach the conclusion about the constant α being equal to $1/\hbar$. Then, Eq. (26) is

$$-\frac{\hbar^2}{2m}\nabla^2 f(\xi) + U(\mathbf{r})f(\xi) = i\frac{\hbar}{2}\frac{df(\xi)}{dt} \tag{27}$$

which differs from the time-dependent Schrödinger equation by only a factor of one-half on the right-hand side. This little difference will be brought into sharp focus in due course. For the time being, however, we observe that Eq. (27) becomes time independent as soon as we repeat the reasoning by using Eq. (23) in place of Eq. (22). As a matter of fact, the final result is

$$-\frac{\hbar^2}{2m}\nabla^2 f(\xi) + U(\mathbf{r})f(\xi) = \hbar\omega f(\xi) \qquad (28)$$

where $\omega = \Omega/2$. Eq. (28) is the well-known time independent Schrödinger equation and, to get back to the canonical time-dependent Schrödinger equation, we can now define a new wave function $\psi(\mathbf{r},t) = f(\xi)exp(-i\omega t)$ that incorporates the arbitrary function $f(\xi)$ we introduced at the very beginning. By doing so, we achieve the correct equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + U(\mathbf{r})\psi(\mathbf{r},t) = i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}. \qquad (29)$$

Nonetheless, the result would be incomplete if we were unable to explain why the factor of one-half makes its appearance on the right-hand side of Eq. (27) and, consequently, in the angular frequency $\omega = \Omega/2$ of Eq. (28). The explanation is rather simple and has some connections to the properties of matter waves (i.e., difference between group velocity and phase velocity in a wave packet).[28] Indeed, going back to the crucial result of Eq. (25) that works for all the points of the trajectory where the momentum is recurrently $\bar{\mathbf{p}}$, we took advantage of $\Omega = \alpha\bar{p}^2/m$ and now, having set $\alpha = 1/\hbar$, it results that $\Omega = \bar{p}^2/(\hbar m)$. But, the particle energy E for which Eq. (25) holds is the classical kinetic energy $E = \bar{p}^2/(2m)$ and, by substitution, we find $\Omega = 2E/\hbar = 2\omega$ where, ultimately, we have defined the new angular frequency according to $E = \hbar\omega$ that coincides with the Planck-Einstein relationship between energy and frequency. In conclusion, we have shown that $\omega = \Omega/2$ corresponds to the correct angular frequency expected in the time-independent Schrödinger equation recovered in Eq. (28) and so popular for its success in the treatment of quantum-mechanical problems.

IV. CONCLUSIONS

To sum up, the correspondence between Newtonian dynamics and Schrödinger picture of quantum mechanics has been examined on the basis of the existence of matter waves dictated by de Broglie relationship. The correspondence revolves around a classical wave equation that describes a function in which the Newtonian information about the dynamical state of the massive particle is stored. The classical wave equation depends on a free parameter that has the same role played by the constant K introduced by Schrödinger in his successful effort to calculate the energy levels of the hydrogen atom. Tuning the parameter on the correct value determined by the de Broglie condition on the momenta, the classical wave equation for conservative forces transforms into the Schrödinger equation.

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