"A call to action": Schrödinger's representation of quantum mechanics via Hamilton's principle

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Abstract

A few years ago, one of the former Editors of this journal launched "a call to action" (E. F. Taylor, Am. J. Phys. **71**, 423 (2003)) for a revision of teaching methods in physics in order to emphasize the importance of the principle of least action. In response, we suggest the use of Hamilton's principle of stationary action to introduce the Schrödinger equation. When considering the geometric interpretation of Hamilton-Jacobi theory, the real part of the action S defines the phase of the wave function $exp(iS/\hbar)$ and requiring the Hamilton-Jacobi wave function to obey wave-front propagation (i.e., $\mathrm{Re}(S)$ is a constant of the motion) yields the Schrödinger equation.

I. INTRODUCTION

The wave mechanics formulation of quantum mechanics relies on the Schrödinger equation and the concept of the wave function.[1] The Schrödinger equation is ordinarily introduced without providing much detail on its conceptual foundations, which leads to some dissatisfaction.[2] Even Schrödinger's original conjecture was heuristic.[3] Its weakness was underlined by Feynman: "some of the arguments he used were even false, but that does not matter; the only important thing is that the ultimate equation gives a correct description of nature".[4] In response, Feynman provided his own derivation of the Schrödinger equation that led him to the path integral formulation of quantum field theory.[5, 6] Responding to the same stimulus and with the aim of a sound derivation, others have come up with various proposals over the last 50 years.[2, 7-23] The approach presented here aims to be both simple and rigorous; it is based on Hamilton's principle of stationary action which students encounter during their studies of Lagrangian and Hamiltonian mechanics.[24] This formulation is conceptually similar to Schrödinger's original proposal based on real-valued Hamilton's functions[3] and answers to "a call to action" by one of the former Editors of this journal emphasizing the action's paramount importance in modern physics.[25] To better contextualize our purpose before going into the details, let us make some preliminary remarks on typical approaches to developing Schrödinger's picture of quantum mechanics.

II. APPROCHES TO DERIVE THE SCHRŐDINGER EQUATION

For context, here we review some familiar approaches to introducing the Schrödinger equation that are found in popular textbooks. The simplest claims that the Schrödinger equation is so well known that its derivation can be neglected in favor of its solutions to fundamental problems.[26, 27] The equation is simply assumed as a fact or presented as a fundamental postulate.[26, 28] Such approaches miss an opportunity to provide insight into the physical meaning of the equation. Other common presentations draw their inspiration from plausibility arguments[29-32] or from linear algebra concepts. These methods have almost nothing in common with the original idea developed by Schrödinger. Within this group, the suggestion to derive the Schrödinger equation from the time-evolution operator is worth mentioning for its close relationship with the Heisenberg representation.[33, 34] Despite their pedagogical usefulness, such proposals have limitations (e.g., the restriction to a free particle, the assumption of a constant potential, the introduction of the composition property for the time-evolution operator) and therefore do not derive the Schrödinger equation in the most general context.

A considerable number of authors have tried to develop an exact formulation from first principles.[2, 7-23] Among these attempts, the use of Hamilton-Jacobi (HJ) theory[24] of classical mechanics has attracted attention because of the guidance it provided Schrödinger in the discovery of his equation. But HJ theory requires the suppression of troublesome nonlinearities (see, for instance, the lengthy procedure in Ref. 2). On the other hand, the HJ approach is advantageous because it references the most important concept in advanced mechanics: the action.[24] Within HJ theory, Hamilton's principal function S is equivalent to the action, and hereafter the two terms are used as synonyms. However, a distinction has to be made between the classical action S_{cl} and its quantum-mechanical counterpart S_{qm}. The former is relevant only in the classical limit $\hbar \to 0$.[29-35] In order to properly describe quantum phenomena, one needs the more general action S_{qm}, which must therefore be a complex-valued function.[36]. This function has the structure of a wave field whose domain includes the nonclassical regions of space where $\mathrm{Im}(S_{qm}) \neq 0$. For instance, if we consider Dirac's suggestion of a wave function of the type $exp(iS_{qm}/\hbar)$ (see Eq. (52) at page 127 of Dirac's textbook[34]) also considered in Bransden and Joachim (see Eqs. (5.398) at page 258 of Ref. 32), agreement between the Schrödinger equation and the HJ equation requires $\hbar \to 0$ (see

Appendix A). Remarkably, the situation changes if we introduce the concept of wave-front propagation of $exp\left(iS_{qm}/\hbar\right)$ in the configuration space. The constraint of wave-front propagation of $exp\left(iS_{qm}/\hbar\right)$ entails a new constant of the motion, i.e., $\mathrm{Re}\left(S_{qm}\right)$, whereas $\mathrm{Im}\left(S_{qm}\right)$ is unconstrained (note that in Dirac's suggestion the entire function S_{qm} is subject to a stationary condition). We will prove that, independent of the actual value of $\hbar$, the classical action is needed to derive the Schrödinger equation. To help the reader gain familiarity with the concept of wave-front propagation in relation to HJ theory, we highlight the reference by Talman[35] and also provide a summary of HJ theory in the next section.

III. BRIEF SUMMARY OF HAMILTON-JACOBI THEORY

Before discussing how to get from the classical formalism to the Schrödinger picture, we summarize the basics of HJ theory. Our notation follows that of Goldstein.[24]

The HJ formulation of classical mechanics has the objective of finding the solution S to the equation

$$H\left(q,\frac{\partial S}{\partial q}\right)+\frac{\partial S}{\partial t}=0 \tag{1}$$

where H is the classical Hamiltonian and S is the Hamilton's so-called principal function (or simply: the action). This function generates the canonical transformation from the n-dimensional set of original generalized variables $\boldsymbol{q}=\{q_1,q_2,\dots q_n\}$ and $\boldsymbol{p}=\nabla S=\{\partial S/\partial q\}=\{\partial S/\partial q_1,\partial S/\partial q_2,\dots\partial S/\partial q_n\}$ to another set of generalized variables $\boldsymbol{Q}=\{Q_1,Q_2,\dots Q_n\}$ and $\boldsymbol{P}=\{P_1,P_2,\dots P_n\}$ where Hamilton's equations of motion are simple to solve. In other words, S is the generating function of the canonical transformation $q_i=q_i(\boldsymbol{Q},\boldsymbol{P},t)$ and $p_i=p_i(\boldsymbol{Q},\boldsymbol{P},t)$ such that the mechanical problem based on the original Hamilton's equations of motion

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \tag{2}$$

reduces to a simpler problem for an analogous set of equations of motion. The simplest set occurs for a constant transformed Hamiltonian K such that

$$\begin{cases} \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \\ \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \end{cases}. \tag{3}$$

Eqs. (3) imply that Q_i and P_i are independent of time and the problem becomes extremely easy to solve. It is possible to prove that the relationship between the old and new Hamiltonians is $K = H + \partial S/\partial t$ and, for the usual choice of $K = 0$, the outcome is the HJ equation Eq. (1) where the substitution of the components of the canonical momentum $p_i = \partial S/\partial q_i$ has been made.[24] The derivatives of S with respect to canonical momenta give $Q_i = \partial S/\partial P_i$ and the inversion of this relationship for each index i completes the canonical transformation $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{P}, t)$ and $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{Q}, \boldsymbol{P}, t)$.

The beauty of the HJ approach appears in the reduction of mechanical (and vectorial) problems with many unknowns to a differential equation whose scalar solution S contains all information about the evolution of the mechanical system under examination. But this elegance is of limited advantage because only a small set of problems can be solved exactly in the HJ context (the harmonic oscillator, Keplerian orbits and a few more). On the other hand, a fundamental property of the action has a more general application: the total time derivative of S is the Lagrangian $L = T - V$ (with T the kinetic energy and V the potential energy)[24]

$$\frac{dS}{dt} = L \tag{4}$$

and therefore, a constant action (i.e., Hamilton's principle) means that the most probable trajectories should keep the time integral of the Lagrangian from varying between the initial and final time of the trajectory. The stationary value of S was also adopted by Feynman in his path integral method - a perfect example of an approach which mixes both classical and quantum ingredients.

Hamilton's principle can be regarded as the guiding star of physics, and plays a central role in classical mechanics[24] as well as the two pillars of contemporary theoretical physics: general relativity (see the Einstein-Hilbert action[37, 38]) and quantum field theory.[39, 40] HJ theory was employed in the early days of quantum mechanics to tackle the puzzling physics of photons and atoms which led to, as an example, the Bohr-Sommerfeld quantization rule. The foundation of wave mechanics through the HJ equation in Eq. (1) begins with Schrödinger's original suggestion[3] to introduce the wave function Ψ through the momentum $\boldsymbol{p} = -i\hbar\nabla\Psi/\Psi$ (note that, in contrast to Schrödinger's use of Eq. (1), the imaginary unit i has been made explicit). Given that the momentum is proportional to the spatial derivative of S, Schrödinger intuited that the variational method based on Hamilton's principle should be reflected in the stationary value of the spatial integration of $|\Psi|^2$ (see the first paper of Ref. [3]). However, Schrödinger originally made a mistake by attaching physical meaning only to the real part of Ψ. The mistake was later corrected in other communications and an interesting account of this episode is available in this journal.[41] If we include both classical and nonclassical regions of S (i.e., spatial regions where S is purely real or has imaginary contributions, respectively), the action is a complex-valued function, and as argued in the next section, it determines Ψ as a propagating wave. Based on this, we show that one can introduce the Schrodinger equation to students without departing from the discoverer's initial proposal to use the action as a fundamental tool. This is then consistent with other well-known approaches (see Appendix B where the Wentzel-Kramers-Brillouin approximation and Feynman's path integral formulation are discussed).

IV. GEOMETRIC INTERPRETATION OF HAMILTON-JACOBI THEORY AND THE SCHRŐDINGER EQUATION

We begin with the introduction of the HJ wave function $\Psi_{\mathrm{HJ}} = exp(iS/\hbar)$ that evolves in configuration space. It was mentioned before that this choice was examined by Dirac but without considering the consequences of a constant value of $\mathrm{Re}(S)$. More recently, the use of Ψ_{HJ} has been popularized in the so-called quantum HJ theory suggested by Leacock and Padgett,[42, 43] which focuses on quantum energy levels.

The introduction of the HJ wave function has several consequences. First of all, due to Ψ_{HJ}'s wave nature, it propagates through spatial regions where S is a complex-valued function. S therefore differs from a purely classical action. It is well known that the opening of non-classical regions of space allows HJ theory to describe matter waves.[44, 45] However, we do not specify the nature of S except that we maintain $\boldsymbol{p} = \nabla S$. Note additionally that the action unit, $\hbar$, in Ψ_{HJ} is without numerical specification and, indeed, the actual value of $\hbar$ does not affect the derivation of the Schrödinger equation. For a Hamiltonian which does not depend explicitly on time, we introduce the Hamilton's characteristic function $W = S + Et$ which is well known to depend on spatial variables and satisfies the reduced HJ equation (i.e., $H(q, \partial W/\partial q) = E$).[24] Since S is complex, the following relationships are valid: $\mathrm{Re}(W) = \mathrm{Re}(S) + Et$ and $\mathrm{Im}(W) = \mathrm{Im}(S)$. After the substitution of $S = W - Et$ into Ψ_{HJ}, the wave function can be factorized according to

$$\Psi_{\mathrm{HJ}} = \psi_{\mathrm{HJ}} exp\left(-\frac{iEt}{\hbar}\right) \tag{5}$$

with $\psi_{\mathrm{HJ}} = exp(iW/\hbar)$. Lastly, another consequence of Dirac's suggestion to use Ψ_{HJ} is the appearance of an imaginary part of $\boldsymbol{p} = \nabla S$. The real part of $\boldsymbol{p}$ is expected to be equivalent to the classical momentum, whereas the possibly non-zero imaginary part of $\boldsymbol{p}$ is expected to play a fundamental role in the quantum-classical transition. An example of how this works for the harmonic oscillator will be given in Section V.

Within the geometric picture of HJ theory, the wave front must satisfy the classical wave equation:

$$\left(\nabla^2 - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\Psi_{\mathrm{HJ}} = 0 \tag{6}$$

where $\boldsymbol{v}_{ph}$ is the phase velocity. This equation encodes the laws of motion of the mechanical system in the motion of the wave Ψ_{HJ} with velocity $\boldsymbol{v}_{ph}$ calculated at $\boldsymbol{q}$. Since the real part of S determines the phase of Ψ_{HJ}, the surface of constant phase (whose points satisfy Hamilton's principle of stationary action) can be found using the differential

$$d[\mathrm{Re}(S)] = \nabla\mathrm{Re}(S)\cdot d\boldsymbol{q} - E dt = 0, \tag{7}$$

where, for time-independent Hamiltonians, $E = -\partial S/\partial t$ has been used. Recalling that the canonical momentum is $\boldsymbol{p} = \nabla S$, Eq. (7) becomes the condition $E = \mathrm{Re}(\boldsymbol{p})\cdot\boldsymbol{v}_{ph}$. In the following, we suppose that $\mathrm{Re}(\boldsymbol{p})$ and $\boldsymbol{v}_{ph}$ are parallel. This very often is the case for purely mechanical systems (however, it is not the case for systems subjected to magnetic forces because the generalized momentum results from the two contributions of the mechanical momentum and the magnetic vector potential). Thus, the time derivative in Eq. (6) can be calculated explicitly thanks to Eq. (5) and, with the result $E = \mathrm{Re}(\boldsymbol{p})\cdot\boldsymbol{v}_{ph}$, the wave equation becomes

$$\nabla^2\psi_{\mathrm{HJ}} + \frac{\mathrm{Re}(\boldsymbol{p})^2}{\hbar^2}\psi_{\mathrm{HJ}} = 0. \tag{8}$$

After the substitution of $\mathrm{Re}(\mathbf{p})^2 = 2m(E - V)$ that defines $\mathrm{Re}(\boldsymbol{p})$ as the classical momentum, we get the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi_{\mathrm{HJ}} + V\psi_{\mathrm{HJ}} = E\psi_{\mathrm{HJ}}. \tag{9}$$

The time-dependent Schrödinger equation is just the outcome of the relationship between ψ_{HJ} and the time derivative of Ψ_{HJ} calculated by means of Eq. (5)

$$\frac{\partial}{\partial t}\Psi_{\mathrm{HJ}} = -i\frac{E}{\hbar}\psi_{\mathrm{HJ}} exp(-iEt/\hbar), \tag{10}$$

which, incorporated in Eq. (9), gives

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_{\mathrm{HJ}} + V\Psi_{\mathrm{HJ}} = i\hbar\,\frac{\partial}{\partial t}\Psi_{\mathrm{HJ}}. \tag{11}$$

Eq. (11) demonstrates our thesis and shows the importance of Hamilton's principle of stationary action in nonrelativistic quantum mechanics. Ultimately, these findings can be easily understood by students aware of Hamilton's principle and its geometric perspective suggested here. As an illustrative example, the transition from the classical wave equation to the Schrödinger equation is demonstrated for the harmonic oscillator in the following section.

One question is yet unanswered. If $\mathrm{Re}(\boldsymbol{p})$ is the classical momentum, how do we interpret $\mathrm{Im}(\boldsymbol{p})$ and its generating function $\mathrm{Im}(S) = \mathrm{Im}(W)$? To this end, let us define $\phi = exp(i\,\mathrm{Re}(W)/\hbar)$ and the amplitude of the wave function $A = exp(-\mathrm{Im}(W)/\hbar)$. This choice allows the factorization of ψ_{HJ} in terms of ϕ and A, or $\psi_{\mathrm{HJ}} = A\phi$. Then, the Laplacian of ψ_{HJ} is

$$\nabla^2\psi_{\mathrm{HJ}} = \phi\nabla^2 A + 2\nabla A\cdot\nabla\phi + \frac{i}{\hbar}\nabla^2\mathrm{Re}(W)\psi_{\mathrm{HJ}} - \frac{1}{\hbar^2}[\nabla\mathrm{Re}(W)]^2\psi_{\mathrm{HJ}} \tag{12}$$

and its use in Eq. (8) yields

$$\phi\nabla^2 A + 2\nabla A\cdot\nabla\phi + \frac{i}{\hbar}\nabla^2\mathrm{Re}(W)\psi_{\mathrm{HJ}} - \frac{1}{\hbar^2}[\nabla\mathrm{Re}(W)]^2\psi_{\mathrm{HJ}} + \frac{\mathrm{Re}(\boldsymbol{p})^2}{\hbar^2}\psi_{\mathrm{HJ}} = 0. \tag{13}$$

But, $\mathrm{Re}(\boldsymbol{p}) = \nabla\mathrm{Re}(W)$ and the last two terms of Eq. (13) cancel each other out. In the end, we get

$$\phi\nabla^2 A + 2\nabla A\cdot\nabla\phi + \frac{i}{\hbar}A\phi\nabla^2\mathrm{Re}(W) = 0. \tag{14}$$

Eq. (14) is a differential equation that establishes a relationship between $\mathrm{Re}(W)$ and $\mathrm{Im}(W)$. In other words, they are not independent from each other as in the case of a generic complex-valued function. The search for $\mathrm{Im}(W)$ and the calculation of corresponding wave function amplitude is illustrated for the fundamental example of the harmonic oscillator in the next section.

V. APPLICATION TO THE HARMONIC OSCILLATOR

The nonclassical character of the wave function Ψ_{HJ} introduced in Section IV depends on the imaginary part of Hamilton's functions W and S ($\mathrm{Im}(S) = \mathrm{Im}(W)$) which dictate the amplitude of ψ_{HJ} (i.e., the spatial component of Ψ_{HJ}). On the other hand, $\mathrm{Re}(S)$ regulates the phase of the wave function which, for the propagating wave front, is subject to the condition given in Eq. (7). In brief, Eqs. (7) and (14) are two coupled differential equations for the real and imaginary parts of W. In what follows, we disregard questions related to the phase. Instead, we dig deep into how the nonclassical character of the complex-valued Hamilton's functions can be elucidated in the case of the harmonic oscillator. To this end, we show how Hamilton's characteristic function of the motionless oscillator (which is purely imaginary) determines the shape of the wave function for the oscillator set in motion.

We begin with a motionless oscillator. No motion implies zero classical energy, or $E_0^{cl} = 0\ E$ and the corresponding characteristic function W_0 satisfies the reduced HJ equation

$$\frac{1}{2m}\left(\frac{d}{dx}W_0\right)^2 + \frac{1}{2}m\omega^2x^2 = 0, \tag{15}$$

where x is the oscillator's displacement from equilibrium. Equation (15) is just a consequence of the general substitutions $E = -\partial S/\partial t$ and $\partial S/\partial x = dW/dx$ in Eq. (1) (note that the total derivative of W is justified for the current one-dimensional problem where W is a function of x only).[24] The solution to Eq. (15) is

$$W_0 = \pm i \frac{1}{2} m\omega x^2. \tag{16}$$

The ambiguity in the sign of W_0 disappears at $x = 0$ when the two alternatives of Eq. (16) are degenerate and real. Since Eq. (15) is valid when the oscillator does not move, this is in perfect agreement with the trivial classical picture of the oscillator. However, this seemingly trivial solution reveals its intriguing consequences when Ψ_{HJ} is allowed to traverse the non-classical regions of space where $x < 0$ and $x > 0$, and W_0 is purely imaginary. In this case, the substitution of the real and imaginary parts of W_0 in $\psi^0_{\mathrm{HJ}} = A_0\phi_0$ with $A_0 = exp(-\mathrm{Im}(W_0)/\hbar)$ and $\phi_0 = exp(i\mathrm{Re}(W_0)/\hbar) = 1$ gives

$$\psi^0_{\mathrm{HJ}} = e^{-m\omega x^2/(2\hbar)} \tag{17}$$

where the choice of the sign of W_0 has been made to ensure convergence at large distances. The result of Eq. (17) is remarkable because, with less effort compared to standard quantum-mechanical procedures, we have produced the ground state wave function.

Next, we examine the quantum harmonic oscillator energy levels $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ with the purpose of understanding their analog in terms of the characteristic function W used here to define the HJ wave function ψ_{HJ}. The scaling of the quantum energy levels demonstrates both a vacuum energy (or zero-point energy), as well as excitations above this labeled by n. The fact that the energy is separable into two independent contributions suggests that we can treat the full state as two noninteracting oscillating "particles" one associated with excitations of energy $n\hbar\omega$ and one associated with the vacuum with energy $\hbar\omega/2$. The analogous physical system within the HJ context of this work generates the factorization of the full wave function into $\psi_{\mathrm{HJ}} = A_0\psi_{\mathrm{W}}$, where ψ_W is associated with excitations above the ground state. Given the relationship between ψ_{HJ} and W, this motivates the introduction of the total Hamilton's characteristic function $W = W_0 + W_{\mathrm{exc}}$ (i.e., the two oscillators are noninteracting). After the factorization of $\psi_{\mathrm{HJ}} = A_0\psi_{\mathrm{W}}$ with $A_0 = exp(iW_0/\hbar)$ and $\psi_{\mathrm{W}} = exp(iW_{\mathrm{exc}}/\hbar)$, and making the obvious simplification to the one-dimensional problem, the classical wave equation in Eq. (6) and $\mathrm{Re}(p)^2 = 2m(E-V)$ yield

$$\frac{d^2A_0}{dx^2}\psi_{\mathrm{W}} + 2\frac{dA_0}{dx}\frac{d\psi_{\mathrm{W}}}{dx} + A_0\frac{d^2\psi_{\mathrm{W}}}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2x^2\right)A_0\psi_{\mathrm{W}} = 0. \qquad (18)$$

The knowledge of A_0 reduces Eq. (18) to

$$\frac{\hbar}{m\omega}\frac{d^2\psi_{\mathrm{W}}}{dx^2} - 2x\frac{d\psi_{\mathrm{W}}}{dx} + \left(\frac{2E}{\hbar\omega} - 1\right)\psi_{\mathrm{W}} = 0 \qquad (19)$$

and, solving for ψ_{W}, the Hermite polynomials $H_n\left(\sqrt{m\omega/\hbar}\,x\right)$ are found for values of E satisfying $E_n = \hbar\omega(n + 1/2)$.[46] The final result for the HJ wave functions is

$$\psi_{\mathrm{HJ}} = e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{m\omega/\hbar}\,x\right) \qquad (20)$$

which, up to a normalization factor, is in perfect agreement with the harmonic oscillator wave functions derived using standard quantum mechanical methods.

Despite the nice conclusion of Eq. (20) one detail is still missing. We should try to understand the energy mismatch between the choice of $E_0^{cl} = 0$ made earlier (see Eq. (15)) and the vacuum energy of $E_0 = \hbar\omega/2$. The mismatch disappears when the nonclassical regions at $x < 0$ and $x > 0$ are part of the wave describing the motionless oscillator. In other words, if E_0^{cl} is the classical energy based on the particle-like understanding of the motionless oscillator, we expect that the wave propagation through the nonclassical regions explains the mismatch. In such an instance, Hamilton's principle applied to the wave-front propagation yields

$$\frac{i}{\hbar}\frac{d^2W_0}{dx^2}\psi_{\mathrm{HJ}}^0 - \frac{1}{\hbar^2}\left(\frac{dW_0}{dx}\right)^2\psi_{\mathrm{HJ}}^0 + \frac{2m}{\hbar^2}\left(E_0 - \frac{1}{2}m\omega^2x^2\right)\psi_{\mathrm{HJ}}^0 = 0, \qquad (21)$$

which simplifies to

$$\frac{1}{2m}\left(\frac{dW_0}{dx}\right)^2+\frac{1}{2}m\omega^2x^2=E_0+\frac{i\hbar}{2m}\frac{d^2W_0}{dx^2}\,. \qquad (22)$$

This is the reduced HJ equation with an extra-term containing the second derivative of W_0. Note that Eq. (22) is analogous to Eq. (A1) in Appendix A generated from the Schrödinger equation. W_0 was found before in Eq. (16) to be $W_0 = im\omega x^2/2$ (after the correct choice of the sign) and the extra-term in Eq. (22) amounts to $-\hbar\omega/2$. It means that the reduced HJ equation is

$$\frac{1}{2m}\left(\frac{dW_0}{dx}\right)^2+\frac{1}{2}m\omega^2x^2=E_0\ -\frac{1}{2}\hbar\omega=E_0^{cl} \qquad (23)$$

so that the choice of the vacuum energy $E_0\ = \hbar\omega/2$ determines the zero-energy level $E_0^{cl} = 0$ used in Eq. (15). In turn, any other value of the energy must be rescaled accordingly as in the eigenvalues of Eq. (19). Note that the addition of a constant value to the energy has no dynamical meaning from the point of view of the fundamental classical laws of motion. All of them (Euler-Lagrangian equations, Hamilton equations, etc.) involve derivatives that cancel the $\hbar\omega/2$ contribution coming from the wave picture summarized in Eq. (6). The energy shift $\hbar\omega/2$ accounts for the fact that a traveling wave cannot be held at rest as in the case of the classical oscillator of Eq. (15). For this reason, any comparison between wave- and particle-like motion should be accompanied by the energy shift of the zero level.

The conclusion of this short example is that HJ wave propagation of Eq. (6) is compatible with Schrödinger picture if the momentum and wave front phase velocity are parallel, and Hamilton's principle is assumed.

VI. CONCLUSIONS

We have shown that the Schrodinger equation can be obtained from Hamilton's principle of stationary action. The stationary action regulates the wave-front propagation of the HJ wave function $exp(iS/\hbar)$. This procedure can be understood by students who have knowledge of Hamilton's principle and classical mechanics.

APPENDIX A: CLASSICAL LIMIT

The classical limit can be readily established after substitution of the wave function $exp(iS/\hbar)$ in the Schrödinger equation. With the help of the canonical relationships $p_i = \partial S/\partial q_i$, the Schrödinger equation reduces to the following

$$\frac{1}{2m}(\nabla W)^2 + V = E + \frac{i\hbar}{2m}\nabla^2 W \tag{A1}$$

where W is again a complex-valued function according to Dirac's suggestion. In the limit of very small $\hbar$, the term involving the Laplacian of W is negligible and Eq. (A1) becomes the HJ equation. However, in the available derivation by Bransden and Joachim[32] (pages 258-260), the Hamilton's functions are tacitly assumed to be real, which goes against Dirac's suggestion. In this case, we would have a wave function with amplitude equal to one, an unnecessary oversimplification. Despite the complex nature of W, the classical limit in Eq. (A1) can all the same be established as long as vanishing $\hbar$ values are accompanied by vanishing $\mathrm{Im}(W)$. In the limit with both $\hbar \to 0$ and $\mathrm{Im}(W) \to 0$, Eq. (A1) can be solved for $\mathrm{Re}(W)$, which is the classical characteristic function giving rise to the classical motion.[24]

APPENDIX B: COMPARISONS WITH OTHER ACTION-BASED METHODS

We conclude our investigation with some comparisons to other popular methods relying on the action and we exclude those attempts where other quantum routes are taken to achieve the objective of the Schrödinger picture. For instance, the method suggested by Green[47] is quite original, but it relies on the Heisenberg representation and the corresponding use of the commutator. Below we discuss two alternative programs: the Wentzel-Kramers-Brillouin (WKB) method and the Feynman's path integral approach.

The WKB program consists of searching for an approximation of the wave function $exp(i\Phi/\hbar)$ that solves the one-dimensional Schrödinger equation.[27, 28, 31-33] Although this method is not concerned with the way the Schrödinger equation is justified, the wave function $exp(i\Phi/\hbar)$ looks very similar to the definition of the HJ wave: $\psi_{\mathrm{HJ}} = exp(iW/\hbar)$. Clearly, to establish the relationship between the two wave functions, we have to gain some insight into how the Φ function relates to the complex Hamilton's characteristic function W. We will see that a difference exists and has decisive implications.

The WKB approach begins with the expansion of the function Φ in powers of $\hbar$ and in the classical limit ($\hbar \rightarrow 0$) the phase of Φ satisfies the HJ equation. The approximation applies when only terms up to $\hbar^2$ are used in the expansion of Φ. When this is done, the approximated wave function depends on the inverse of the square root of the linear momentum p

$$\psi_{\mathrm{WKB}} \propto \frac{1}{\sqrt{p}} exp(\pm iW_{cl}/\hbar) \tag{B1}$$

and a more general solution is built on the linear combination of the propagating and counter-propagating waves generated by the choice of the sign in Eq. (B1). Notably, the classical Hamilton's characteristic function W_{cl} appears in the WKB wave function where W_{cl} is a real function in contrast to the complex W of ψ_{HJ}. This fact explains why ψ_{WKB} has singular behavior at the turning points where $p = 0$ whereas ψ_{HJ} of Section IV is regular.

Now, we turn to the Feynman's path integral approach. Again, we simplify to a one-dimensional mechanical problem where the path integral approach to the Schrödinger representation "is easily interpreted physically as the expression of Huygens' principle for matter waves".[5] This amounts to calculating the wave function at time $t + \epsilon$ according to

$$\Psi_{\mathrm{F}}(x, t+\epsilon) \propto \int exp(iS(x,y)/\hbar)\Psi_{\mathrm{F}}(y,t)dy \tag{B2}$$

where $S(x, y)$ coincides with Hamilton's principal function in the classical limit and Ψ_{F} describes the wave front characterizing the Huygens' surface of constant phase. Expansion with respect to the time parameter ϵ leads to an approximated result on both sides of Eq. (B2) (the integral is broken into several contributions arising from the infinitesimal time steps of duration ϵ connecting x and y). Indeed, recalling Eq. (4), a small change S is expected for the time variation ϵ during which the mechanical system goes from x to y. Then, the action changes by $S(x, y) = \epsilon L_{\epsilon}$. In addition to the factor ϵ, another parametric dependence appears in the Lagrangian L_{ϵ} as a function of coordinate y and velocity $(y - x)/\epsilon$. Based on this, the expansions of both sides of Eq. (15) give linear terms in ϵ and, in turn, their equality makes it possible to sort out the time-independent Schrödinger equation.

In this very short summary of the Feynman's approach to the Schrödinger equation, it is still clear that all the ingredients of the HJ approach of Section IV (wave-front propagation and Hamilton's principle) are present here too. Both methods are rooted in the tools of classical mechanics which acquire a new meaning when they are extended to the quantum realm. As Feynman said "there is a pleasure in recognizing old things from a new point of view".[5]

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